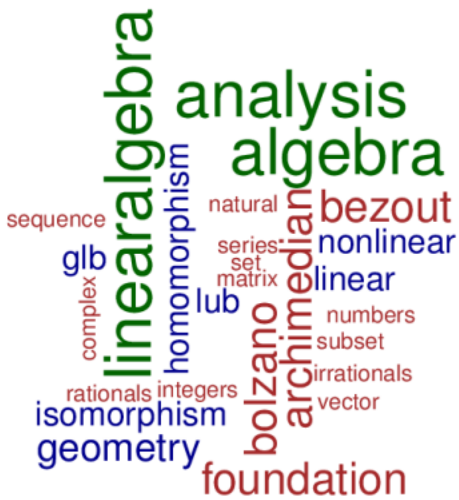
nurture

Aims:

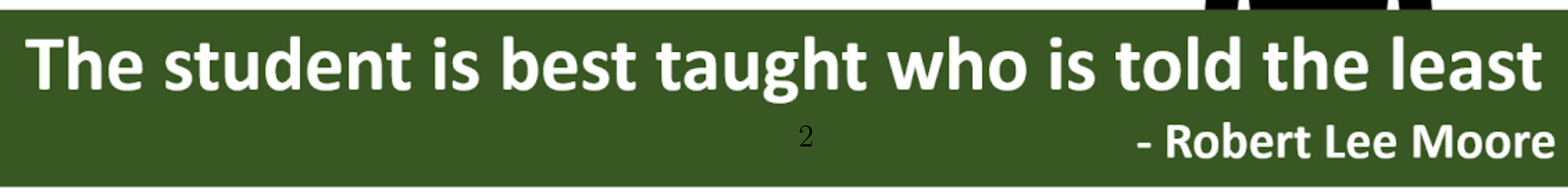
- To provide an opportunity to the students who have interest in Mathematics, to nurture what their future career might demand, namely (to list a few)
  - Axiomatic view of the subject
  - How to conventionally go about the proofs?
  - Nurture a set of concepts in detail such that bridging the students to higher level courses
- **Attitude:** Motivate and demonstrate to the students to be more open minded, flexible in thinking, guess and conquer the results such that the students gain confidence in doing mathematics independently
- **Values:** To facilitate students to build a professional network with a diverse group of committed students and faculties from pan India which will help them to be a valuable team player for the country

Technical details about the workshop

- There will be four interactive sessions everyday on Foundation, Analysis, Algebra and Linear Algebra
- The workshop will start with an ice breaker session and there will be a few more sessions on soft skills everyday
- Students will be given thinking and writting exercise everyday
- There will be special lectures by eminent scientist on the contribution of famous Indian Mathematicians namely S S Pillai and S Minakshisundaram
- A short course on Graph theory is also expected
- Students Seminor: A few students may be allowed to give seminor



Workshop Cloud



nurture

Summer School in Mathematics to the memory of S S Pillai

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(in alphabetical order)

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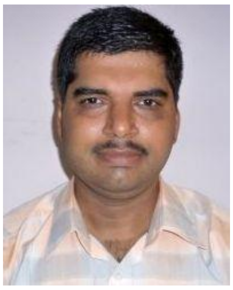

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If you cant explain it simply, you don’t understand it well enough.

- Albert Einstein

Date: 14th, 15th and 16th August 2016(tentative)

Title of the talk: **Contributions of S S Pillai**

Abstract: A reasonably detailed discussion of SS Pillai's work will be presented. His contribution to Waring's problem was described in 1950 by K S Chandrashekaran as, " almost certainly his best piece of work and one of the very best achievements in Indian Mathematics since Ramanujan." The plan is to cover both his professional and personal life.

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Date: 20th & 21st August 2016 (tentative)

Title of the talk: **Life and works of S. Minakshisundaram**

Abstract: S. Minakshisundaram is one of the world class mathematicians of the colonial India who have made long lasting impact on Indian Mathematics. Along with Ananda Rau, R. Vaidyanathaswamy, T. Vijayaraghavan, S. Chawla, Hansraj Gupta and S.S. Pillai, he was instrumental in developing Indian Mathematics. He was one of the pioneers in India who worked on partial differential equations. His celebrated work with the Swedish mathematician A. Pleijel has played an important role in the proof of Patodi-Singer index theorem. In this talk we plan to briefly describe his life and works.

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List of Selected Participants

#	Id	Name	category	College/University/Institute name
1	2016RSM001	D. Kelvin Gandhi	III B.Sc.	A.V.V.M Sri Pushpam College(Autonomous),Tamilnadu
2	2016RSM002	L Daniel Vijay	II B.Sc.	Jamal Mohamed College,Tamilnadu
3	2016RSM003	Kabileshkumar K	III B.Sc.	Sacred Heart College, Tamilnadu
4	2016RSM004	Muniyasamy	III B.Sc.	American College, Tamilnadu
5	2016RSM005	Evin Jayson	II B.Sc.	Christ College Irinjalakuda, Kerala
6	2016RSM006	Piyush Baghel	II B.Sc.	Deshbandhu College, New Delhi
7	2016RSM007	Devendrapuri Nenpuri Goswami	II B.Sc.	K C College, Churchgate, Mumbai
8	2016RSM008	Muthineni Ashok	II B.Sc.	Nagarjuna Government College (AUTONOMUS) Nalgonda, Telangana
9	2016RSM010	Baaavaraj Ramling Manurkar	III B.Sc.	S P College, Maharastra
10	2016RSM011	Abin George	II B.Sc.	St Thomas College Palai, Kerala
11	2016RSM012	Augus Kurian	III B.Sc.	St. Berchmans College, Changanasserry, Kerala
12	2016RSM013	Bijesh Kunnath Sukumaran	III B.Sc.	St. Xavier's College, Mumbai
13	2016RSM014	Anitha. A	III B.Sc.	Alagappa Government Arts College, Karaikudi, Tamilnadu
14	2016RSM015	Pavithralakshmi M	III B.Sc.	Ayya Nadar Janaki Ammal College, Tamilnadu
15	2016RSM017	Anjana. N. Menon	II B.Sc.	Chirst College ,Thrissur, Kerala
16	2016RSM018	A Rajalakshmi	II B.Sc.	Dr Ngp Arts And Science College Coimbatore. Tamilnadu
17	2016RSM019	Anumeena S	II B.Sc.	Dr Ngp Arts And Science College Coimbatore. Tamilnadu
18	2016RSM020	Gurvir Kaur	III B.Sc.	Guru Nanak National College, Doraha,Ludhiana, Punjab
19	2016RSM021	Sowndharya.V	III B.Sc.	Idhaya College For Women, Kumbakonam. Tamilnadu
20	2016RSM022	Rajam S	III B.Sc.	Jairam Arts & Science College, Tamilnadu
21	2016RSM024	MADHUMATHI D	II B.Sc.	KONGU ARTS AND SCIENCE COLLEGE,ERODE, Tamilnadu
22	2016RSM025	SARANYA N	II B.Sc.	NANDHA ARTS AND SCIENCE COLLEGE,ERODE, Tamilnadu
23	2016RSM026	NAGARATHINAM V	III B.Sc.	NANDHA ATRS AND SCIENCE COLLEGE,ERODE, Tamilnadu
24	2016RSM027	R. LATHA MAHESWARI	III B.Sc.	NMS S VELLAICHAMY NADAR COLLEGE, Tamilnadu
25	2016RSM028	ARCHANA S	III IMSc	PONDICHERRY UNIVERSITY, Pondicherry
26	2016RSM029	Janci Rani.P	II B.Sc.	PSG college of arts and Science, Tamilnadu

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27	2016RSM030	S.Dhivya Priya	II B.Sc.	Sri Sarada College For Women(Autonomous), Tamilnadu
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29	2016RSM032	Jisna Mary P J	II B.Sc.	St Alberts College, Ernakulam, Kerala.
30	2016RSM033	B.Madhurima	II B.Sc.	St.Francis College For Women, Telangana
31	2016RSM034	Susmita Tripathy	II B.Sc.	St.Francis College For Women, Telangana
32	2016RSM035	K.Karthika Devi	III B.Sc.	St.Joseph College,Trichy, Tamilnadu
33	2016RSM036	Athira Johny	III B.Sc.	Union Christian College, Aluva, Kerala
34	2016RSM037	Keerthi M.V.	III B.Sc.	Union Christian College, Aluva, Kerala
35	2016RSM038	M. Pavithra	III B.Sc.	Vellalar College For Women,(Autonomous), Tamilnadu
36	2016RSM040	Gayathri.V	III B.Sc.	Vivekanandha College Of Arts And Sciences For Women, Tamilnadu
37	2016RSM041	G.Muthulakshmi	III B.Sc.	Vivekanandha College Of Arts And Sciences For Women, Tamilnadu
38	2016RSM042	Prabha V S	II B.Sc.	Vysya College /PERIYAR University, Tamilnadu
39	2016RSM043	R.Olivarasi	III B.Sc.	Virudhunagar Hindhu Nadars Senthikumara Nadar College In Virudhunagar

List of Local Participants

#	Id	Name	category	College/University/Institute name
1	2016RSM044	Karthikeyan S	III B.Sc.	Thiruvalluvar Government Arts College, Rasipuram
2	2016RSM045	Gopi Krishna K	III B.Sc.	Thiruvalluvar Government Arts College, Rasipuram
3	2016RSM046	Sivaranjani N	III IMSc	Central University Of Tamil Nadu
4	2016RSM047	Divya A	III IMSc	Central University Of Tamil Nadu
5	2016RSM048	Vasanthakumari M	III IMSc	Central University Of Tamil Nadu

Teacher / Scholar Participants

#	Id	Name	category	College/University/Institute name
1	2016RSMT1	Mohanappriya G	Scholar	Kongunadu Arts And Science College
2	2016RSMT2	Prithvi M	Scholar	Central University Of Tamil Nadu
3	2016RSMT3	Priyanga B	Scholar	Central University Of Tamil Nadu
4	2016RSMT4	Thatchaayini R	Scholar	Central University Of Tamil Nadu

Subbayya Sivasankaranarayana Pillai

Born : April 5, 1901
Death : August 31, 1950

An Outstanding Indian Number Theorist

by

Thangadurai R (HRI Allahabad)

"...The audience may be a little disappointed at the scanty reference to Indian work. ... However, we need not feel dejected. Real research in India started only after 1910 and India has produced Ramanujan and Raman." This was the statement of Dr. S. Sivasankara-narayana Pillai in the 36th Annual session of the Indian Science Congress on 3rd January, 1949 at Allahabad university. Of Course, now India has produced many more legends. Indeed, he himself is one such. He was well known for his contributions in the theory of numbers, in particular, in the Warning's problem.

Life History of S.S. Pillai:

On April 5, 1901 S. Sivasankaranarayana Pillai (as S. S. Pillai - known to Mathematical community) was born at Vallam near the famous Water falls town namely Courtallam which is situated in Tirunelveli district, Tamilnadu. His parents were Subbayya Pillai and Gomati Ammal. His mother died a year after his birth. He grew up under the care of an old relative in the family. When S. S. Pillai was five years old, a master was arranged to teach Pillai at his house. At the age of nine, Pillai joined a Middle School at Shencottah. There, by looking at his high intellectual ability and his potential, a teacher called 'Sastriar' gave a constant support and encouragements. Then for his Matriculation course, he moved to a local High School. During this period, he was shattered by his father's sudden death and he was struggling to continue his high school education because of monetary reasons. Fortunately, his former teacher 'Sastriar' came to rescue with monetary support to complete not only to his school education but also to go beyond. He went to Nagercoil to do his Intermediate course in the Scott Christian College with a Scholarship. Then he did B. A at Trivandrum in the Maharaja's college.

In 1927, S. S. Pillai received a Research Studentship at the University of Madras to work under Professors K. Ananda Rao and R. Vaidyanathaswamy. Then he joined Annamalai university as a lecturer. There his mathematical achievements seem to have peaked. Indeed, Madras university honored him by awarding D. Sc. for his research achievements. He was the first one to get this honor in Mathematics. He shifted to University of Travancore in 1941. Then a year later, he joined Calcutta University as a lecturer.

For his achievements, he was invited to visit the Institute of Advance Studies, Princeton, USA for a year. Also, he was invited to participate in the International Congress of Mathematicians at Harvard University as a delegate of Madras University. So, he proceeded to

USA by air in the august 1950. But due to the air crash near Cairo on August 31, 1950, Indian Mathematical Community lost one of the best known mathematicians. Of course, his mathematical works will continue to be cherished by mathematicians for all time to come.

Some Pearls Form His Mathematical History:

A good part of his mathematical achievements was done in Annamalai University. He worked mostly in analytical number theory. One of his famous contributions to the solutions of Waring' s Problem. Waring's problem states as follows; "Let k be a given positive integer. Then $g(k)$ defined to be the smallest positive integer such that any natural number n can be written as sum of at most $g(k)$ - k-th powers of integers. What is the exact value of $g(k)$?".

For instance, when $k = 2$, we have 1, 4, 9, 16, 25, $\cdots$, m^2, $(m+1)^2$, $\cdots$ squares of integers. Suppose the denominations of our currency (may be, rupee notes) in these numbers. For example, if we want to give Rs. 103 to a shop keeper, then, we can give him in the following denominations, $\underbrace{1, 1, \cdots, 1}_{103 \text{ times}}$ or $\underbrace{4, 4, \cdots, 4}_{25 \text{ times}}, 1, 1, 1$ or $1, 1, 1, 25, 25, 25, 25$ or $1, 1, 1, 36, 64$ or $1, 4, 49, 49$. The Warning's problem, in this case, is asking for the minimum number of denominations required to cover all the positive integers. In our example, we see that the minimum number is 4. In fact, Lagrange proved that $g(2) = 4$ in 1770. Note that for some integers, we can manage with fewer number of squares (that is, in stead of 4, we may manage with 3 or 2 or 1). For instance, we have $36 = 6^2$ (rather all the squares). From the work of D. Hilbert (1909) it is known that $g(k)$ exists for every $k \geq 2$ and it is finite. Also, it is known that $g(3) = 9$.
In 1935, S. S. Pillai proved the Waring's problem for $k \geq 6$ by proving that $g(k) = 2^k + \ell - 2$ where ℓ is the largest natural number $\leq (3/2)^k$. Also, Pillai proved that $g(6) = 73$ by computing, in this case, the precise value of ℓ as defined above. Then to complete the history of this problem $g(5) = 37$ was proved by J. Chen in 1965. Then the remaining only case was $g(4) = 19$ and it was proved by R. Balasubramanian along with M. Deshouillers and F. Dress in 1987.

S. S. Pillai worked on Diophantine approximations too and he proved the following beautiful theorem; "Let a, b, m and n be natural numbers and $\delta > 0$. Then for any integral x and y if we have $am^x - bn^y \neq 0$, then we have

$$|am^x - bn^y| > m^{(1-\delta)x}.$$

for all $x \gg 0$". Based on this theorem, S. S. Pillai proposed a problem on the finiteness of the number of integer solutions for an 'exponential' Diophantine equation. This is known as S. S. Pillai's conjecture. This conjecture is yet to be resolved.

S. S. Pillai gave a new proof of Bertrand's Postulate; " Given any $x \geq 2$, there is always a prime number p in between x and $2x$". Indeed, S. Ramanujan also gave a different proof of this result. The best elementary proof known for this result is due to P. Erdos.

Below we shall list all the research publications of Pillai:

1. On $\nu(k)$, Proc. Indian Acad. Sci., IX (1935), 175 (added May, (1951)).
2. On the equation $2^x - 3^y = 2^X + 3^Y$, Bull. Calcutta Math. Soc. 37, (1945), 15-20.
3. Highly composite numbers of the tth order, J. Indian Math. Soc. (N. S.), 8, (1944), 61-74.
4. Correction to my paper "Bertrand's postulate", Bull. Calcutta Math.
5. Soc., 37, (1945), 27. Correction, Ibid, 37 (1944), 27. On m consecutive integers. IV, Bull. Calcutta Math. Soc., 36, (1944). 99-101.
6. On $a^X - b^Y - b^y a^x$, J. Indian Math. Soc. (N. S.) 8, (1944), 10-13.
7. On Waring's problem with powers of primes, J. Indian Math. Soc. (N.S.) 8, (1944), 18-20.
8. On the smallest primitive root of a prime, J. Indian Math. Soc. (N.S.) 8, (1944), 14-17.
9. Highly abundant numbers, Bull. Calcutta Math. Soc., 35, (1943), 141-156.
10. Lattice points in a right-angled triangle - III, Proc. Indian Acad. Sci., Sect. A, 17, (1943), 62-65.
11. Lattice points in a right-angled triangle - II, Proc. Indian Acad. Sci., Sect. A, 17, (1943), 58-61.
12. On $\sigma_{-1}(n)$ and $\phi(n)$, Proc. Indian Acad. Sci., Sect. A, 17, (1943), 67-70.
13. On a congruence property of the divisor function, J. Indian Math. Soc. (N. S.) 6, (1942), 118-119.
14. On the divisors of $a^n + 1$, J. Indian Math. Soc. (N. S.), 6, (1942), 120-121.
15. On a problem in Diophantine approximation, Proc. Indian Acad. Sci., Sect. A, 15, (1942), 177-189.
16. On algebraic irrationals, Proc. Indian Acad. Sci., Sect. A, 15, (1942), 173-176.
17. (with George, A), On numbers of the form $2^a 3^b$ - II, Proc. Indian Acad. Sci., Sect. A, 15, (1942), 133-134.
18. On numbers of the form $2^a 3^b$ - I, Proc. Indian Acad. Sci., Sect. A, 15, (1942), 128-132.
19. Nurnberg proof of Cauchy's general principle of convergence, Math. Student, X (1942), 91-92.
20. On the definition of oscillation, Math. Student, IX (1941), 165-167.

21. On the sum function connected with primitive roots. Proc. Indian Acad. Sci., Sect. A, 13, (1941), 526-529.

22. On m consecutive integers - III, Proc. Indian Acad. Sci., Sect. A, 13, (1941), 530-533.

23. On Waring's problem $g(6) = 73$, Proc. Indian Acad. Sci., Sect. A, 12 (1940), 30-40.

24. On Waring's problem with powers of primes, Proc. Indian Acad. Sci., Sect. A, 12, (1940), 202-204.

25. Waring's problem with indices $\geq n$, Proc. Indian Acad. Sci., Sect. A, 12, (1940), 41-45.

26. A note on Gupta's previous paper, Proc. Indian Acad. Sci., Sect. A, 12, (1940), 63-64.

27. On a linear Diophantine equation, Proc. Indian Acad. Sci., Sect. A, 12, (1940), 199-201.

28. On normal numbers, Proc. Indian Acad. Sci., Sect. A, 12, (1940), 179-184.

29. Generalisation of a theorem of Mangoldt, Proc. Indian Acad. Sci., Sect. A, 11, (1940), 13-20.

30. On m consecutive integers - II, Proc. Indian Acad. Sci., Sect. A, 11, (1940), 73-80.

31. On m consecutive integers - I, Proc. Indian Acad. Sci., Sect. A, 11, (1940), 6-12.

32. On the converse of Fermat's theorem, Math. Student, VIII (1940), 132-133.

33. Symposium on Waring's problem, Chairman's address, Math. Student, VII (1939), 165-168.

34. On Stirling's approximation, Math. Student, VII (1939), 70-71.

35. On Waring's problem, IX (on universal Waring's problem with prime powers), Jour. Indian Math. Soc., N. S. III (1939), 221-225.

36. On Waring's problem, VIII (with polynomial summonds), Jour. Indian Math. Soc., N. S. III (1939), 205-220.

37. On numbers which are not multiples of any other in the set, Proc. Indian Acad. Sci., Sect. A, 10, (1939), 392-394.

38. On the number of representations of a number as the sum of the square of a prime and a squarefree integer, Proc. Indian Acad. Sci., Sect. A, 10, (1939), 390-391.

39. A note on the paper of Sambasiva Rao, J. Indian Math. Soc. 3, (1939), 266-267.

40. On the smallest prime of the form $km+l$, Proc. Indian Acad. Sci., Sect. A. 10, (1939). 388-389.

41. On normal numbers. Proc. Indian Acad. Sci., Sect. A. 10, (1939), 13-15.

42. On Waring's problem with powers of primes, Proc. Indian Acad. Sci., IX (1939), 29-34.

43. On the addition of residue classes, Proc. Indian Acad. Sci., VII (1938), 1-4.

44. Generalization of a theorem of Davenport on the addition of residue classes, Proc. Indian Acad. Sci., VI (1937), 179-180.

45. On Waring's problem VI, Annamalai University Journal, (1937), 171-197.

46. (with S. Chowla), The number of representations of a number as a sum of n non-negative nth powers, Quart. J. Math. Oxford Ser., 7 (1936), 56-59.

47. (with S. Chowla), Hypothesis K of Hardy and Littlewood, Math. Z., 41 (1936), 537-540.

48. On Waring's problem V : on $g(6)$, Jour. Indian Math. Soc., New series, II (1936), 213-214.

49. On $a^x - b^y = c$, Jour. Indian Math. Soc., New series, II (1936), 119-122. Correction, Ibid, p, 215.

50. On the set of square free numbers, Jour. Indian Math. Soc., New series, II (1936), 116-118.

51. On Waring's problem II, Jour. Indian Math. Soc., New series, II (1936), 16-44.

52. On Waring's problem IV, Annamalai University Journal, VI, 1 (1936), 54-64.

53. On Waring's problem III, Annamalai University Journal, VI, 1 (1936), 50-53.

54. On Waring's problem I, Annamalai University Journal, V, 2 (1935), 145-166.

55. Periodic simple continued fractions, Annamalai University Journal, IV, 2, (1935), 216-225.

56. On the nature of conduct of conics $S = 0$ and $S + \lambda T = 0$, Math. Student, III (1935), 156.

57. Remarks on Dr. Moessner's paper, Math. Student, II (1934), 104-106.

58. On a passtime common among South Indian school children, Math. Student, I (1933), 52-56.

59. On an arithmetic function, Annamalai University Journal, II, 2 (1933), 242-248.

60. On the sum function of the number of prime factors of N, J. Indian Math. Soc., XX (1931), 70-87.

61. On an arithmetic function concerning primes, Annamalai University Journal, I, 2 (1932), 1-9.

62. On the indeterminant equation $x^y - y^x = 0$, Annamalai University Journal, I, 1 (1932), 59-61.

63. An old result concerning ϕ function, J. Indian Math. Soc., XIX (1931), 165-168.

64. (with S. Chowla), Periodic simple continued fractions, J. London. Math. Soc., 6 (1931), 85-89.

65. On the inequality $0 < a^x - b^y \leq n$, J. Indian Math. Soc., XIX (1931), 1-11.

66. On some Diophantine equations, J. Indian Math. Soc., XVIII (1930), 291-295.

67. On a function analogues to $G'(k)$, J. Indian Math. Soc., XVIII (1930), 289-290.

68. On the numbers which contain no prime factors of the form $p(kp+1)$, J. Indian Math. Soc., XVIII (1930), 51-57.

69. (with S. Chowla), On the error tems in some asymptotic formulae in the theory of numbers - II, J. Indian. Math. Soc., 18 (1930), 181-184.

70. (with S. Chowla), On the error tems in some asymptotic formulae in the theory of numbers - II, J. London. Math. Soc., 5 (1930), 95-101.

71. A theorem concerning the primitive periods of integer matrices, J. London Math. Soc., 4 (1929), 250-251.

72. On the representation of a number as the sum of two positive powers, J. London Math. Soc., 3 (1928), 56-61.

73. On a function connected with $\phi(n)$, Bull. Amer. Math. Soc., XXXV (1929), 837-841.

74. On some functions connected with $\phi(n)$, Bull. Amer. Math. Soc., XXXV (1929), 832-836.

75. On some empirical theorems of Scherk, J. Indian Math. Soc., XVII (1927-28), 164-171.

76. A test for groups of primes, J. Indian Math. Soc., XVII (1927-28), 85-88.

Obituary Columns

1. Chandrasekharan, K. Obituary: S. S. Pillai. J. Indian Math. Soc. (N.S.) Part A. 15, (1951). 1-10

2. S. Raghavan, Outstanding Indian Mathematician - Dr. S. S. Pillai, Ramanujan News letters, (2001)

S. S. Pillai Contributions

by
Thangadurai R (HRI Allahabad)

Waring's problem

A discussion of Pillai's mathematical work must start with Waring's problem and vice versa! However, since this has been written about in detail in the June 2004 issue, we mention this problem in passing and refer the readers to the above-mentioned Resonance article by C.S.Yogananda.

Waring's problem asks for the smallest number $g(k)$ corresponding to any $k \geq 2$ such that every positive integer is a sum of $g(k)$ numbers each of which is the k-th power of a whole number. Hilbert had shown that such a finite number $g(k)$ does exist. The ideal Waring's conjecture predicts a particular value of $g(k)$. Indeed, if 3^k is divided by 2^k, the quotient is $[(3/2)^k]$, and some remainder r, where $[t]$ denotes the greatest integer less than or equal to t. Now, the number

$$2^k[(3./2)^k] - 1 = ([(3/2)^k] - 1)2^k + (2^k - 1)1^k$$

is a sum of $2^k + [(3/2)^k] - 2$ numbers which are k-th powers and is not the sum of a smaller number of k-th powers. Hence,

$$g(k) \geq 2^k + [(3/2)^k] - 2$$

This ideal Waring conjecture asserts that this lower bound is the correct bound also. Pillai proved, among other things, that this ideal Waring conjecture holds good under the condition on k that the remainder r on dividing 3^k by 2^k satisfies $r \leq 2^k - [(3/2)^k] - 2$ (this is known to hold for all $k \leq 471600000$). At this time, the ideal Waring conjecture is known to hold for all large enough k.

Problem on consecutive numbers

Pillai proved in 1940 that any set of n consecutive positive integers where $n \leq 16$, contains an integer which is relatively prime to all the others. However, there are infinitely many sets of 17 consecutive integers where the above fact fails. For instance, $N+2184, N+2185, \cdots, N+2200$ is such a set whenever N is a multiple of $2.3.5.7.11.13 = 30,030$.

How spread-out are perfect powers?

Look at the sequence of all perfect powers of positive integers:

$$1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, \cdots$$

We observe that differences between consecutive terms can be: 1 $(9-8)$, 2 $(27 = 25)$, 3 $(4-1)$, 4 $(36-32)$, 5 $(32-27)$ etc.

Pillai conjectured that consecutive terms can be arbitrarily far apart. In other words, given any number, one can find consecutive terms whose difference is larger than that given number. Equivalently, the conjecture asserts:

Given a positive integer k, the equation $x^p - y^q = k$ has only finitely many solutions in positive integers $x, y, p, q \geq 2$.

This is unproved as yet even for one value of $k > 1$ although it is known now that if one of these 4 parameters is fixed, the finiteness holds.

Average of Euler's ϕ-function

Euler's ϕ-function, denoted by $\phi(n)$ is an arithmetic function defined on natural numbers that counts the number of natural numbers $1 \leq m \leq n$ with $(m, n) = 1$. Euler gave a formula which can be proved using inclusion - exclusion principle as follows.

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

where the product varies over all the distinct prime divisors. This formula shows that the functional value fluctuates a lot.

In analytic number theory, to study such fluctuating arithmetical functions, one often looks at their *average behaviour*. One can prove that the average value from 1 to x is $\frac{3}{\pi^2}x$ but the interesting part is to have an idea of the error which would be introduced if we take this value. In anlytic number theory, this methodology of '*determining the main term and estimating the error term*' is fundamental because we cannot deduce anything concrete if the error term is of the same order as the main term! One has

$$\sum_{1 \leq n \leq x} \phi(n) = \frac{3}{\pi^2}x^2 + E(x)$$

where $E(x)$ is the remainder or error term in the average.

Dirichlet showed that for any given $\epsilon > 0$, there is a constant $C > 0$ so that $|E(x)| \leq Cx^{1+\epsilon}$ for all $x > 0$. Later, this was improved to $|E(x)| \leq C'x \log x$ for all $x > 0$ by Mertens.

Sylvester prepared a table of values for $\sum_{n\leq x}\phi(n)$ and $\frac{3}{\pi^2}x^2$ for all $x = 1, 2, \cdots, 1000$. However, he failed to notice that $E(820) < 0$ and made a conjecture that $E(x) \geq 0$ for all x. In 1929, Chowla wrote a letter to Pillai where he predicted that $E(x) > 0$ for infinitely many values of x and $E(x) < 0$ for infinitely many values of x.

In order to prove an error term, say $|E(x)| \leq cg(x)$, is tight for some non-negative function $g(x)$, one needs to produce a positive constant c_0 and infinitely many x's such that $|E(x)| > c_0 g(x)$.

Such a result is called an *'omega'-result* in analytic number theory; *we write* $E(x) = \Omega(g(x))$.

Chowla and Pillai showed that $E(x) = \Omega(x \log\log\log x)$.
Such a result took many years to generalize. There is a conjecture by Montgomery that

$$E(x) = O(x \log\log x) \text{ and } E(x) = \Omega_{\pm}(x \log\log x),$$

which is still open.

A variant of Tic-Tac-Toe!

In 1933, Pillai studied a variant of Tic-Tac-Toe game as follows. Let $n \geq 3$ be an integer and $t \leq n$ be another integer. Suppose $n \times n$ grid with n^2 squares is given in the plane. Let P and Q be two players playing. By turns each mark a square. The rule of playing the game is as follows. Suppose P is the starter and, let P_r and Q_r be the squares marked respectively by P and Q during their r-th turns. After P's (or Q's) s-th turns, whoever marked t squares in a straight line wins the game.

Pillai proves that when $t = n$ and the game is carefully played, then it will end always in a draw. However, if $t < n$, then for a given t, there is a function $f(n)$ depending on n such that if $t \geq f(n)$, then the game ends in a draw. When $t < f(n)$, he proved that the player who starts will win. Also, he proved that $f(n) \leq n + 1 - \sqrt{n/6}$ and $f(n) = n$ for all $n = 3, 4, 5$, and 6. For large values of n, the correct order of $f(n)$ is still unknown!

Smooth Numbers

Smooth numbers are numbers which have only 'small' prime factors. For example, 1,620 has prime factorization 22 34 5; therefore 1,620 is 5-smooth because none of its prime factors is greater than 5. Smooth numbers have a number of applications to cryptography. For example, the very smooth hash functions are used constructively to get a provably secure design. They also play a role in music theory apparently (Longuet-Higgins, H. C. (1962), "Letter to a musical friend", Music Review (August): 244248)!
For other applications, the interested reader may also consult A.Granville's article 'Smooth

numbers: Computational number theory and beyond' in the Proceedings of an MSRI workshop, 2004 and A.Hildebrand and G. Tenenbaum's article: *Integers without large prime factors*, J. Theor. Nombres Bordeaux, **5** (1993), no. 2, 411-484.

For any real numbers $x, y > 1$ with $y \leq x$, we define $\psi(x, y)$ to be the number of positive integers $t \leq x$ such that if a prime $p|t$, then $p \leq y$. In other words, $\psi(x, y)$ counts all the y-smooth numbers up to x. Ramanujan (in a letter to Hardy) was the first to study these smooth numbers when $y = 3$(!)

He obtained a nice asymptotic formula for $\psi(x, 3)$. In the 7-th conference of Indian Mathematical Society during 3-5, April, 1931 at Trivandrum, Pillai extended the above result of Ramanujan by obtaining a result which the following result which implies an asymptotic formula for $\psi(x, y)$ if $y > 1$ is a fixed real number. This is technical to state but we mention it here in passing, for the interested reader:

If $p_1, p_2, \cdots, p_r \leq y$ are all the prime numbers less than y, then

$(\psi(x,y) - \frac{(\log x)^r}{r! \prod_{i=1}^r \log p_i} + \frac{(\log x)^{r-1} \log(p_1 \cdots p_r)}{2(r-1)! \prod_{i=1}^r \log p_i}) / \log^{r-1} x \to 0$ when $x \to \infty$.

Around that time, Dickman obtained an asymptotic result for $\psi(y^u, y)$ for any fixed $u > 0$. The word 'asymptotic' here refers to an assertion of the form 'what is the limit of $\frac{\psi(y^u,y)}{y^u}$ as $u \to \infty$'?
A more rigorous proof of Dickman's result, in modern standards, was supplied by Chowla and T. Vijayaraghavan in 1947 where they used the above result of Pillai!

It should be mentioned that Ramanujan - see page 337 of S. Ramanujan, *The lost notebook and other unpublished papers*. Narosa Publishing House, (1988) - had the following entry. We write in the standard notations as above:

$$\psi(x, x^c) \sim x \left(1 - \int_c^1 \frac{du}{u}\right) \text{ if } 1/2 \leq c \leq 1;$$

$$\psi(x, x^c) \sim x \left(1 - \int_c^1 \frac{du}{u} + \int_c^{1/2} \frac{dv}{v} \int_v^{1-v} \frac{du}{u}\right) \text{ if } 1/3 \leq c \leq 1/2;$$

$$\psi(x, x^c) \sim x \left(1 - \int_c^1 \frac{du}{u} + \int_c^{1/2} \frac{dv}{v} \int_v^{1-v} \frac{du}{u} - \int_c^{1/3} \frac{dz}{z} \int_z^{(1-z)/2} \frac{dv}{v} \int_v^{1-v} \frac{du}{u}\right)$$

if $1/4 \leq c \leq 1/3$*; and so on.*
This is nothing else than Dickman's asymptotic formula for $\psi(x, y)$!

The Story of Two Peerless Indian Mathematicians

S Chowla and S S Pillai

by
B Sury (ISI Bangalore), R Thangadurai(HRI Allahabad)

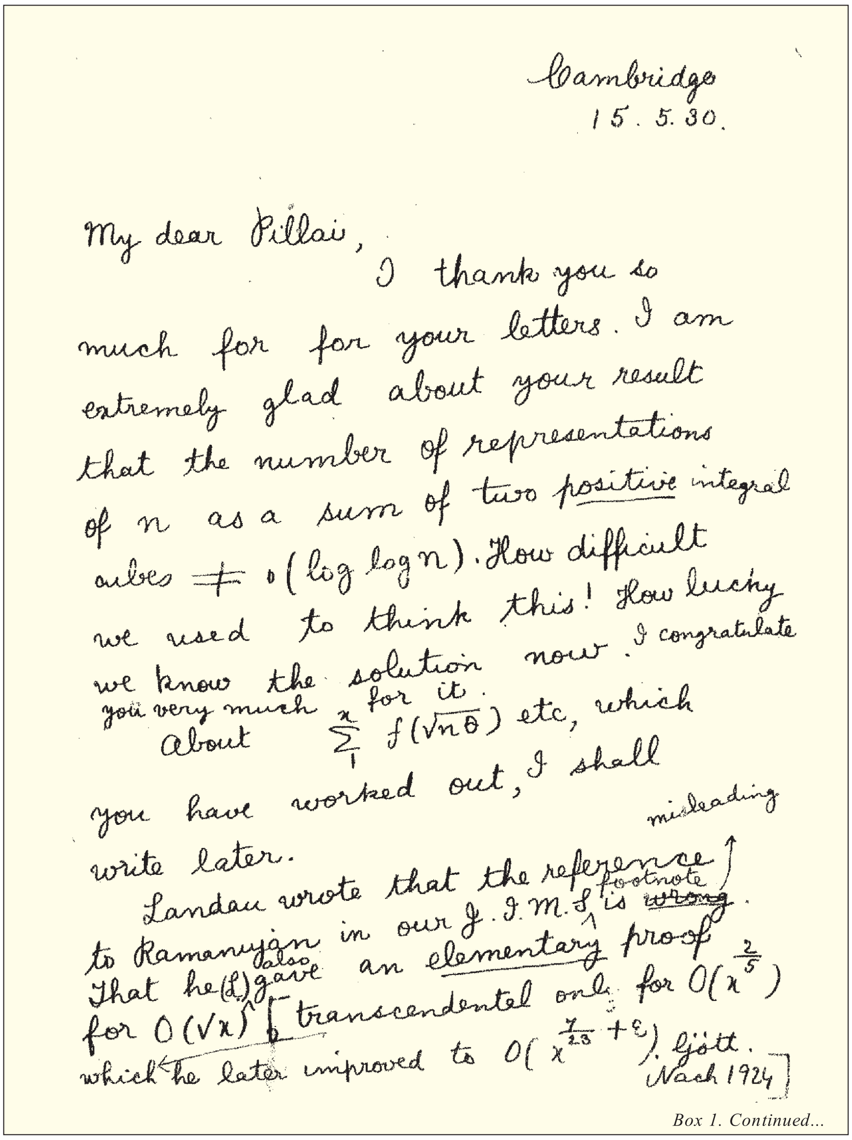
Ramanujan's story has been well-chronicled and it is well-known that generations of mathematicians in India have been inspired by it. However, not much has been written about the period immediately following Ramanujan's. Two of the most famous Indian mathematicians of this period are Sarvadaman Chowla and S S Pillai. We journey through some of the very interesting and illuminating correspondences between Chowla and Pillai. We also attempt to convey some of the beauty and depth of their mathematical work.

Introduction

Sarvadaman Chowla (1907-1995) and S S Pillai (1901-1950) were two of the foremost mathematicians to emerge from India in the generation immediately after Ramanujan. The Mathematics Genealogy Project lists both Ramanujan and Chowla among the students of Littlewood! This article specially features Chowla and Pillai. Thejournal Resonance had already featured Pillai; however, a discussion of Chowla is necessarily intertwined with one of Pillai. It has been mentioned by G H Hardy that after Ramanujan, the greatest Indian mathematician was Pillai. We journey through some of the very interesting and illuminating correspondences between Chowla and Pillai which reveals also other personal and historical aspects. Apart from that, we discuss Chowla's and Pillai's mathematical works. We select only those topics which are more elementary or easy to describe while conveying some of the beauty and depth of the ideas. Fortunately, in the works of Chowla and Pillai,we can nd a veritable treasury which is accessible at a level that can be enjoyed by even the non-expert. Each of their works has an element of surprise and an element of elegance and simplicity. They worked on a wide spectrum of areas of number theory. While discussing their proofs, we attempt to retain as much of the original ideas in the arguments as possible. It is an enigma that even a layman may ask a question in elementary number theory which turns out to be non-trivial. The fact that several old problems in elementary number theory remain unsolved to this day has been referred to in different ways by people. To quote Professor K Ramachandra, *"figurative terms, what has been solved can be likened to an egg-shell, and what remains to be solved to the in nite space surrounding it."*

Chowla Pillai Correspondence

Starting in the late 1920s, and up to one month before Pillai's demise in 1950, Chowla and Pillai maintained a regular correspondence. Interestingly, in the earliest available letter dated 8th of January, 1929, Chowla men- tions among other things that the number 175,95,9000 is the smallest integer that can be expressed as sum of two positive cubes in three different ways. He goes on to express the hope that now they can their proper work". They published joint papers starting in 1930 with a famous piece of work on the Euler's totient function. Some other themes that they collaborated on were concerned with solutions to the BrahmaguptaPell equation and the Waring problem. The correspondence between these two stalwarts is mathematically illumi- nating to read.



Cambridge
15. 5. 30.

My dear Pillai,
I thank you so much for for your letters. I am extremely glad about your result that the number of representations of n as a sum of two positive integral cubes $\neq o(\log \log n)$. How difficult we used to think this! How lucky we know the solution now. I congratulate you very much for it.
About $\sum_1^x f(\sqrt{n}\theta)$ etc, which you have worked out, I shall write later.
Landau wrote that the reference footnote to Ramanujan in our J. I. M. S. is ~~wrong~~ misleading. That he (L) also gave an elementary proof for $O(\sqrt{x})$ [transcendental only for $O(x^{\frac{2}{5}})$ which he later improved to $O(x^{\frac{7}{23}+\varepsilon})$ Gött. Nach 1924]

Box 1. Continued...

Box 1. Continued...

I am sorry for my mistake.

If there any misprints in our J. London M. S. paper please write to me. I could not find any I hope there are ~~a~~not too many faults in it. Please tell them to me.

The pages ~~letter~~ I sent you on Waring's theorem were ~~was~~ for you. It has appeared in J. L. M. S. I am sorry I did not write to you before. But I hope Mr. Narasinga Rao will excuse me if they are also going to publish it.

Walfisz sent some matter that proves that

$$\sum_{1}^{R} E(x) = \frac{3}{2\pi^2} R^2 + O(R^{\frac{5}{3}});$$

can you improve it? I hope I shall soon send his proof.

(I have been doing Landau's 1912 Gött. Nach. paper lately, ~~so I had nothing~~ I hope you have in Madras Univ. Library)

The correspondence also reveals the intellectual honesty they possessed and the joy each drew from the other's successes. One of the letters written by Chowla after he joined St.Stephens College in Delhi expresses his reluctance to be a coauthor of some result where he felthe had not contributed enough. Through the years, he expresses almost in every letter his gladness for the correspondence between them!

The number theorist K Ramachandra spoke of his rst meeting with Chowla at the Institute for Advanced Study in Princeton during the former's rst visit there. After discussing mathematics, Chowla got them both bottles of 'pepsi' from a vending machine. After the meeting, Ramachandra says that he ran around the premises muttering that he drank pepsi with Chowla!

Chowla's fertile imagination earned him the sobriquet of 'poet of mathematics' from his associates. Chowla passed away in the US in 1995 at the age of 88. On the other hand, Pillai died tragically at the age of 49 in 1950. Pillai was invited to visit the Institute for Advanced Study in Princeton for a year. The flight which he boarded to participate in the 1950 International Congress of Mathematicians tragically crashed near Cairo on the 31st of August.

Least Prime Quadratic Residue

Chowla's lifelong pre-occupation with class number of binary quadratic forms led him to discover some rare gems on the way, so to speak! An interesting problem, useful in cryptography, for instance, is to nd for a given prime p, the smallest prime q which is a quadratic residue (that is, a square) modulo p. For example, the quadratic reciprocity law tells us that if $p \equiv \pm 1$modulo 8, then 2 is the least quadratic residue mod p. proved the following beautiful result:

Theorem: *Let $p > 3$ be a prime such that $p \equiv 3 \mod 8$. Let $l(p)$ denote the least prime which is a quadratic residue modp. If the number h(-p) of classes of binary quadratic forms of discriminant -p is at least 2, then $l(p) < p = 3$. If $h(-p) = 1$, then $l(p) = (p+1) = 4$ (and, therefore, $(p+1) = 4$ is prime!).*

Remarks

- The theorem implies, in particular, that for primes $p > 3$; $p \equiv 3 \mod 8$, we have $l(p) = (p+1) = 4$ if and only if $h(-p) = 1$, because $p = 3 < (p+1) = 4$ for $p > 3$.
- The proof of the theorem is easy and uses Minkowski's reduction theory of quadratic forms which produces in each equivalence class of positive-de nite forms, a unique one $ax^2 + bxy + cy^2$ which is 'reduced' in the sense that $|b| \leq a \leq c$.

Chowla's Counter-Examples to a Claim of Ramanujan and a Disproof of Chowla's Conjecture

Among Ramanujan's numerous astonishing results, there are also occasional lapses. One such was his 'proof' (in his very rst paper of 1911) that the numerators of Bernoulli numbers are primes. This is false; for instance, denoting by B_n the Bernoulli number defined by

$$\frac{Z}{e^z-1} = \sum_{n\geq 0} B_n \frac{Z^n}{n!}$$

and by N_n , the numerator of $B_n = n$, the numbers $N_{20}; N_{37}$ are composite. In 1930, Chowla showed that Ramanujan's claim has infinitely many counter-examples. Surprisingly, Chowla returns to this problem 56 years later(!) in a joint paper with his daughter and poses as an unsolved problem that N_n is always square-free. In a recent article, Dinesh Thakur pointed out that Chowla's question has in nitely many negative answers by showing: *For any fixed irregular prime p less than 163 million, and any arbitrarily large k, there exists a positive integer n such that* N_n *is divisible by* p^k

If we observe (from the existing tables) that 37^2 divides N_{284} , Chowla's question has a negative answer. The proof of the more general assertion uses the so-called Kummer congruences which essentially assert that the value of

$$\frac{(p^{n-1}-1)B_n}{n} \bmod p^k$$

depends (for even n) only on n modulo $p^{k-1}(p-1)$, if p-1 does not divide n. Using this as well as certain functions called p-adic L-functions, the general assertion of arbitrarily large powers can also be obtained.

For further contributions of Chowla please visit the link http://www.isibang.ac.in/~sury/chowlapillai.pdf

S. Minakshisundaram

Born : 12th October 1913
Death : 13th August 1968

A Tribute to the Memory of SMS

(by Smt.M.Parvathi. (Wife of SMS))

Dr.S.Minakshisundaram married me on May 10th 1937. I was born in 1924 and was only 13 years of age . He was loved and respected by all the family members, even distant relations. His values were sincere and divine. He cared and showed great regard to one and all in the family. He struggled from early life and achieved fame, name and wealth. He gave generously any help, monetary or otherwise to any relations or family member empathetically. Any poor student came to his door spontaneously he helped . He would pay their fees and for hostel accommodation and food. His death brought so many unknown people to our home to express their sorrow, pay homage to a kind, generous and great man.

He had great love and respect for his mother. His childhood, education till college was all in Madras. In 1937 he was given a small job in the madras Univerisity Library. Smultaneously he started Research work. He was working hor his Doctoral thesis. Along with him Sri R.V. Vaidyanath also was working. He went with Madras Univ. Prof, Sri Ananda Rao to Anna Malai Univ. to deliver a lecture. Then he was appointed as Tutor in Madras Loyola.

In 1940 he was awarded the Doctorate D.Sc. he was hardly 27 years of age. He joined Andhra University which was at Guntur in those days because of Evacuation from Waltair during the IInd world war. In 1946 along with Sri. Chandra Shekhar he went to America for two years. He was invited to princeton Institute. He worked with several stalwarts in the field of Mathematics and Physics such as Albert Einstein. In 1948 returned by ship via Ceylon. He wrote a letter that I should take his brothers help and proceed by boat to Colombo. He wanted some money also to be brought for the official clearance. As our youngest child,a girl of 2 years could not stay without me, we had to carry her.

Rerurning to madras we proceded to Visakhapatnam. The war was over and the Andhra University was shifted back to Waltair from Guntur. He re-joined as Reader. In 1950 he became Professor. Again from US, he was invites to give guest lecturers and for conference. He also was offered appointments from IIT madras and at Tirupathi. The Vice- Chancellor of Andhra University requested him not to accept these offers. Even Osmania University had offered him a Professors post.

Dr. S. Radha krishnan, the second President of India, his son Dr. S. Gopal, the Historian and professor in delhi University started his career as lecturer in Andhra University. He was a close friend of my husband. Along with them was Dr. S.T. Krishnama Chari, professor of German and French. The three were good friends.

Again in 1957 he visited US and Edinburgh U.K. for guest lecturers. In 1966 he went to Simla as Research Professor. He was there for a year or so when the V.C. of A.U. Dr. K.R. Srinivas iyengar invited him to start the P.G. Centre at Guntur as special officer and Principal. The heat and work load affected his health. He had a heart attack and could not work much. He spent a few with his mother in Madras our older daughter als lived there. he spent a few days in her home too. He returned to Vizag and took VRS. The V.C. and the UGC Chairman Kothari visited us. We shifted into our own house. By August 13, 1968 he had attained moksha.

A Tribute to the memory of SMS

(by K. Girija (daughter of SMS))

My father was born in Trichur, Kerala in 1913. He was the blessing and boon after several years of penance performed by my grandparents. They had visited several holy places, temples and had taken a vow in Guruvayur Temple of Krishna. My grandmother though married by the age of eight years as a child she connceived at the age of 24. In those days it was considered very late and probably she had to suffer the ill social talk also. Naturally the birth of the boy child brought great joy. Being Saivites the child was named Minakshisundaram. He was called Jeja by the fond parents . Jeja meaning God. The vow at Guruvayur was fulfilled. The infant was taken to Guruvayur placed on the large weighing scales. Equal weight of butter was offered to the deity of the temple.

Girl children are most loved and petted in the family. Till the age of eight, he had long hair. My grandmother would comb and braid his hair. In the traditional way the religious ritual of Upanayanam was performed at the age of 7. With the thread ceremony the long braid became a Namboodiri Kudumi. Living in Kerala for several generations Malayalam was the language spoken; yet Telugu was spoken at home in the adulterated fashion of Malayalam, Tamil and Telugu. The life style, food habits and religious rituals had the Malayalam and Tamil influences in the family. My father had no sisters.

My father was educated in Madras. His S.S.L.C. education was from C.R.C. High School, Perambur, Madras. His two younger brothers were educated in the Ramakrishna Mutt School, Mambalam, Chennai. Education in that school was free and in the next street from their aunts home. After school my uncles did not opt for higher education due to monetary difficulties. Both looked for small jobs. My father with his brilliance was able to get monetary assistance and scholarships for further education. Naturally though the amount was paltry he lived and pursued in higher education within the means happily. He cared for his father who was too ill to move. He would help his mother in the household chores. If his mother was ill or followed traditional ways he would cook and wash clothes of his parents.

Their life style was economically backward, traditional south Indian -Dakshinadhi- Brahmin Community. Since my father was born in Tirchur and spent his childhood in Kerala, they spoke Malayalam at home . Shifting to Chennapatnam-Madras- Chennai- because of his fathers job in the BritishRaj, Tamil became the mother tongue of the family. so to say. But the original ancestors hail from Andhra and shifted to the deep south . Telugu is basically the mother-tongue and arva-telugu was and is dominant in the family. In fact till recent times ancestral landed property was in Peddapuram near Samalkot. Verdant, fertile, flourishing, prosperous fields which the middle men looked after. Independent India and Land Ceiling Act helped them to grab the lands.The family was not aware, even though my father and one uncle migrated back to Andhra.

To read full tribute follow the link
http://minakshisundaram.org/tributes/daughters-tribute-by-girija/

S Minakshi Sundaram (SMS): A Glimpse into his life and work

by
Thangavelu S (IISc Bangalore)

During the period 1900-1950 India witnessed the emergence of several pure mathematicians such as K Ananda Rau (1893-1966), R Vaidyanathaswamy (1894-1960), T Vijayaraghavan (1902-1955), S S Pillai (1901-1950) and S. Minakshisundaram (1913-1968). With the excption of Vaidyanathaswamy who studied logic, set theory and general topology, all the rest were first class analysts. This is not at all surprising given the fact that both Ananda Rau and Vijayaraghavan were students of G H Hardy anf the other two studied with Ananda Rau. In this article we would like to introduce Minakshisundaram to the readers of Resonance and give a brief summary of his work. For more details about his life and personality we refer the readers to the obituary written by K.G. Ramanathan (8). Indeed, we have drawn generously from this article.

Subbaramiah Minakshisundaram was born in Trichur, Kerala on october 12,1913. His father was originally from Salem, Tamilnadu (not far away from the birth place of Ramanujan) and so Minakshisundaram had all his education in Madras. He took his BA (Hons.) in Mathematics from Loyala College in 1934 securing a first class in the Madras University examination. Though there was always a strong tradition of scholarship and learning in and around Madras, many brilliant young men of that time used to opt for the more lucrative and prestigious administrative services. But young Minakshisundaram was diffrent-he joined Madras University as a research scholar and started working with Ananda Rau.

After taking the DSc degree from Madras University in 1940, Minakshisundaram found himself without job. Thanks to the timely help of Fr. Racine who was professor at Loyola College, he could earn a living by coaching students for the universiy examinations. During these years he and Fr. Racine organised a weekly Mathematics Seminar which attacted many enthusiastic participants like K Chandrasekharan and K.G. Ramanathan. Fortunately he got the job of a lecture at Andhra University in Waltair.

In 1944, Marshall H Stone was in Madras and he wanted to meet the best young mathematicians there, especially Minakshisundaram and Chandrasekharan. M.H. Stone, who was a proffessor at Chicago, had a reputation for discovering young talents and shapong their career. For both Minakshisundaram and Chandrasekharan this meeting with Stone was turning point. By the efforts of Stone they were offered a membership at the Institute for Advanced Study in Princeton. The atmosphere at the Institute gave him a great boost and it was there where Minakshisundarams best mathematical works were done. He collaborated with the Swedish mathematician Ake Pleijel and wrote his most quoted paper.

Minakshisundaram returned to india in 1948 by which time he was Internationally recognized for his brilliant work. Early in 1950 Andhra Univeristy promoted him to full proffessorship in Mathematical Physics. Soon after he became a professor, he spent a few months at the Tata institute of Fundamental Research, Bombay where he collaborated with K. Chandrasekharan in writing of the monograph Typical means. (K. Chandrasekharan was brought to India in 1949 by Homi Bhabha to build the School of Mathematics in TIFR which he did with great success). Minakshisundaram made a couple of brief visits to the US and in 1958 went to Edinburgh to give a half-hour lecture on Hilbert algebras at the International Congress of Mathematicians

Minakshisundaram collaborated with Ake Pleijel, K. Chandrasekharan, O Szasz and C.T. Rajagopal. He also had several research students working with him but unfortunately most of them gave up research after taking their PhD. He tried hard to generate enthusiasm in his students and colleagues for mathematical research. But he found the atmosphere in an Indian University stifling for creative work. He longed for a place like Princeton where he could work without any hindrance. He was appointed a Professor at the newly created Institute for Advanced Studies in Simla but by then his health was deteriorating after a bad heart attack. He passed away on August 13, 1968.

we now proceed to describe some aspects of Minakshisundarams work As this article is meant for Resonance readers, we are not in a position to go beyond an outlline of his various results. Interested readers can go through the papers mentioned under Suggested Reading. A complete list of papers of Minakshisundaram is given at the end of (8).

Minakshisundaram started his mathematical career by working on Tauberian theorems and summability results of classical Fourier analysis. Then under the influence of M.R. Siddiqui of Osmania University, Hyderabad he studied non-linear equations of parabolic and hyperbolic type. His work on partial differential equations formed part of his doctoral dissertation entitled Fourier Ansatz and non-linear parabolic equations. An important outcome of his thesis is a long series of papers on eigenfunction expansions associated to boundary value problems. He then investigated the associated Zeta function using heat kernel culminating in his famouse paper with Ake Pleijel (5).

In order to appreciate the results of Minakshisundaram on non-linear parabolic equations let us begin by considering the linear equation.

$$\frac{\partial}{\partial t}u(x,t)-\frac{\partial^2}{\partial x^2}u(x,t)=0, u(x,0)=f(x) \quad (1)$$

with the boundary condition $u(0,t)=u(\pi,t)=0$ for all $t \geq 0$. Note that for each natural number n the function $u_n(x,t)=e^{-n^2t}\sin nx$ satisfies the equation with $u_n(x,0)=\sin nx$. If the initial condition f(x) can be expanded as

$$f(x)=\sum_{n=1}^{\infty} c_n \sin nx \quad (2)$$

then a formal solution of (1) is given by

$$u(x,t)=\sum_{n=1}^{\infty} c_n e^{-n^2t} \sin nx \quad (3)$$

This method is well known and goes back to D Bernoulli (for the case of the wave equation) around 1750. In 1807, Joseph Fourier asserted that any f with $f(0)=f(\pi)=0$ can be expanded as in (2) and gave the formula (3) for the solution of (1). With this investigation mathematics saw the birth of Fourier series which led to the development of analysis.

Since the work of Minakshisundaram depends heavily on 'heat kernels' let us spend some more time on the above equation (1). The constants c_n appearing in (2) are given by Fourier's formula

$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \quad (4)$$

and hence the formal solution u in (3) can be written as

$$u(x,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \left(\int_0^{\pi} f(y) \sin ny \, dy \right) \sin nx. \tag{5}$$

Defining a function $K(x,y,t), t > 0$ by the equation

$$K(x,y,t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-n^2 t} \sin nx \sin ny$$

we can represent the solution u as

$$u(x,t) = \int_0^{\pi} K(x,y,t) f(y) dy. \tag{6}$$

The function $K(x,y,t)$ is called the heat kernel for the operator $\frac{d^2}{dx^2}$.

Consider now the inhomogeneous equation

$$\frac{\partial}{\partial t} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = g(x,t). \tag{7}$$

A solutin of this problem is given by

$$u(x,t) = \int_0^{\pi}\int_0^{t} K(x,y,t-s)g(y,s)ds\,dy \tag{8}$$

as can be be easily checked. But things become more and more complicated if g also depends on u and u_x. In his work, Minakshisundaram addressed the problem.

$$\frac{\partial}{\partial t}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = g(x,t,u,u_x) \tag{9}$$

Assuming that g, as a function of x, can be expanded in terms of sin nx, Minakshisundaram looked for a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin nx \tag{10}$$

This leads to an infinite system of nonlinear integral equations which Minakshisundaram solved by appealing to a fixed point theorem of Schauder thus proving the existnece and in some cases uniqueness of solutions of nonlinear requations of parabolic type.

The methods used in his thesis gave rise to a new kind of summability, which he called the Bessel summability. A detailed study of this was taken up by K Chandrasekharan in 1942 (then an MSc student) and much later by M S Rangachari (1965). Another important outcome is his work on generalised Fourier expansions which we describe now. In solving the heat equation (1) we have made use of the functions $\phi_n(x) = \sin nx$ which are eigenfunctins of the boundary value problem

$$\frac{d^2}{dx^2}u(x) = \lambda\, u(x), u(0) = u(\pi) = 0 \tag{11}$$

Consider an analogue of this problem in two dimensions. Suppose Ω is a bounded connected open subset of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. Let $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ be the Laplacian and consider the Dirichlet problem

$$\Delta u = -\lambda^2 u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega. \tag{12}$$

By spectral theory of self adjoint operators there exists a sequence $\lambda_n \geq 0, \lambda_n \to \infty$ of eigevalues with corresponding eigenfunctions $\phi_n(x), x = (x_1, x_2)$. That is, we have

$$\Delta\phi_n(x) = -\lambda^2{}_n\phi_n(x), x \in \Omega, \phi_n(x) = 0, x \in \phi\Omega \qquad (13)$$

The eigenfunctions ϕ_n can be normalised so that

$$\int_\Omega |\phi_n(x)|^2 dx = 1$$

Given such a sequence of eigenfunctions ϕ_n associated to the boundary value problem (12) one is tempted to solve the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \; in \, \Omega, u(x,t) = 0 \; on \, \partial\Omega \qquad (14)$$

for t > 0 with the initial condition $u(x,0) = f(x)$ by assuming a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda^2_n t} \phi_n(x) \qquad (15)$$

This is clearly a solution of (14) but the initial condition will be satisfied only if we have an expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \qquad (16)$$

Thus we are naturally led to the eigenfunction expansion (16) which is sometimes called generalised Fourier expansion of *f*.

Now we can ask questions similar to those one asks concerning classical Fourier series. The above expansion (16) with

$$c_n = \int_\Omega f(x)\phi_n(x)dx$$

Returning to the eigenfunctions ϕ_n associated to the boundary value problem (12) consider the following 'Zeta function'

$$Z(s)=\sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(y)}{\lambda_n^{2s}} \tag{17}$$

In a series of paper Minakshisundaram started investigating properties of this function. His aim was to determine the asymptotic distribution of the eigenvalues λ^2_n and the eigenfunctions ϕ_n H Weyl, T Carleman and A Pleijel had done some pioneering work on these problems. The novelty of the approach taken by Minakshisundaram lies in the fact that he used the fundamental solution of the heat equation given by

$$K(x,y,t)=\sum_{n=1}^{\infty}e^{-\lambda_n^2 t}\phi_n(x)\phi_n(y) \tag{18}$$

converges inFormula - Thangavelu() but for other modes of convergence (eg. pointwise) one has to assume further conditions on f or appeal to some summability methods (eg. Riesz means). This is the case even with the classical Fourier series. In a series of papers Minakshisundaram proved: (i) uniqueness theorems for the above expansions (ii) Riesz summability and (iii) uniform converges for a class of functions.

Apart from studying generalised Fourier expansions, Minakshisundaram also proved several interesting on classical multiple Fourier series. In a joint work with K. Chandrasekharan he studied the (Bochner-) Riesz means associated to double Fourier series. In a earlier work with O Szasz he studied the absolute convergence of multiple Fourier series. As remarked by S. Bochner in his reviews of these papers, the conditions on the function involve properties of its spherical means. This is quite remarkable since the interplay between the regulairy of sperical means and pointwise convergence of Fourier series has been studied recently by M. Pinsky and others.

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$$\lambda_n^2 \text{ and the eigenfunctions } \phi_n$$

H. Weyl, T. Carleman and A Pleijel had done some pioneering work on these problems. The novelty of the appoach taken by Minakshisundaram lies in the fact that he used the fundamental solution of the heat equation given by

$$K(x,y,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \phi_n(x)\phi_n(y)$$

in studying these problems. Using properties of the heat kernel K(x,y,t) he was able to generalise the asymptotic results of Carleman.

The Swedish mathematicians A. Pleijel (who,incidently, was son-in-law of M.Riesz) was visiting the Institute at the same time as Minakshisundaram. They started a fruitful collaboration which resulted in a very influential paper (5). In this, they studied the Zeta function associated to the Laplace-Beltrami operator on a compact Riemannian manifold without boundary. Using heat kernel methods they were able to prove that the series

$$Z(s) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{\lambda_n^{2s}}$$

(where now $\Delta\delta_n = -\lambda_n^2 \delta_n$ on *M*) can be analytically continued to the whole *s*-plane as an entire function of s with zeros at non-positive integers when $x \neq y$ and as a meromorphic function when $x = y$. This led to the asyptotic behaviour

$$\sum_{\lambda_n^2 < \lambda} \phi_n(x)^2 \sim \frac{\lambda^{\frac{d}{2}}}{(2\sqrt{\pi})^d \Gamma(\frac{d}{2}+1)},$$

where d is the dimension of *M*. Integrating over *M* one obtains for $N(\lambda)$, the number or eigenvalues less than λ, the asymptotic behavriour

$$N(\lambda) \sim \frac{c_0 \lambda^{\frac{d}{2}}}{(2\sqrt{\pi})^d \Gamma(\frac{d}{2}+1)}$$

where Co is the volume of M. The ideas used in this paper turned out to be very useful in studying geometric properties of manifolds using analytic techniques. An interesting problem is to investigate how the eigenvalues

$$\lambda_n^2$$

as its fundamental tones, is it possible to figure out the shape of the drum by simply listening to the music it produces? in 1964, Minor showed that the eigenvalues cannot distinguish isometric manifolds. The idea of Minakshisundaram and Pleijel was to look at the heat kernel

$$K(x,y,t) = \sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\phi_n(x)\phi_n(y)$$

along the diagonal when t is small. For the Fourier series, that is for the manifold S^1 (the unit circle in $\mathbb{R}^2$),

$$K(x,x,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{-n^2 t}$$

which is a theta function, usually denoted by $\theta(t)$. In view of Jacobi's identity we have

$$K(x,x,t) = (4\pi t)^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi^2 n^2}{t}}$$

from which we obtai the asymptotic expansion

$$K(x,x,t) \sim (4\pi t)^{-\frac{1}{2}}\{1+0(1)\}$$

for small t. Minakshisundaram and Pleijel established such an expansion in the general case. For small t they showed that the kernel defined in (19) satisfies

$$K(x,x,t) \sim (4\pi t)^{-\frac{d}{2}}\left\{1+k_1(x)t+k_2(x)t^2+\dots\right\} \qquad (2)$$

The coefficients $k_i(x)$ are now called Minakshisundaram coefficients.

The computation and geometric interpretatin of Minakshisundaram coefficients is still an open problem. When $d = 2$, Mc Kean and Singer showed that k_1 and k_2 are related to the curvature properties of the manifold. Integrating (20) over M we get

$$\sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \sim (4\pi t)^{-\frac{d}{2}} c_0\left\{1+a_1 t+a_2 t^2+\dots\right\} \qquad (21)$$

which shows that the dimension d and the volume c_0 of the manifold can be 'heard' from the spectrum. For the action of the Laplace-Beltrami operator on p forms, $0 \le p \le d$ one has a similar expansion

$$\sum_{n=1}^{\infty} e^{-\lambda_{n,p}^2 t} \sim (4\pi t)^{-\frac{d}{2}} c_0\left\{\binom{d}{p}+a_{1,p}t+a_{2,p}t^2+\dots\right\}$$

In a 1970 paper V.K. Patodi of TIFR computed the coefficients a1,p and a2, p for all p. Thus the idea of using heat kernel has proved to be very fruitful and extremely powerful. It has led to the work of Atiyah, Bott and Patodi on a new approach to the index theorem for elliptic operations.

Suggested Reading:

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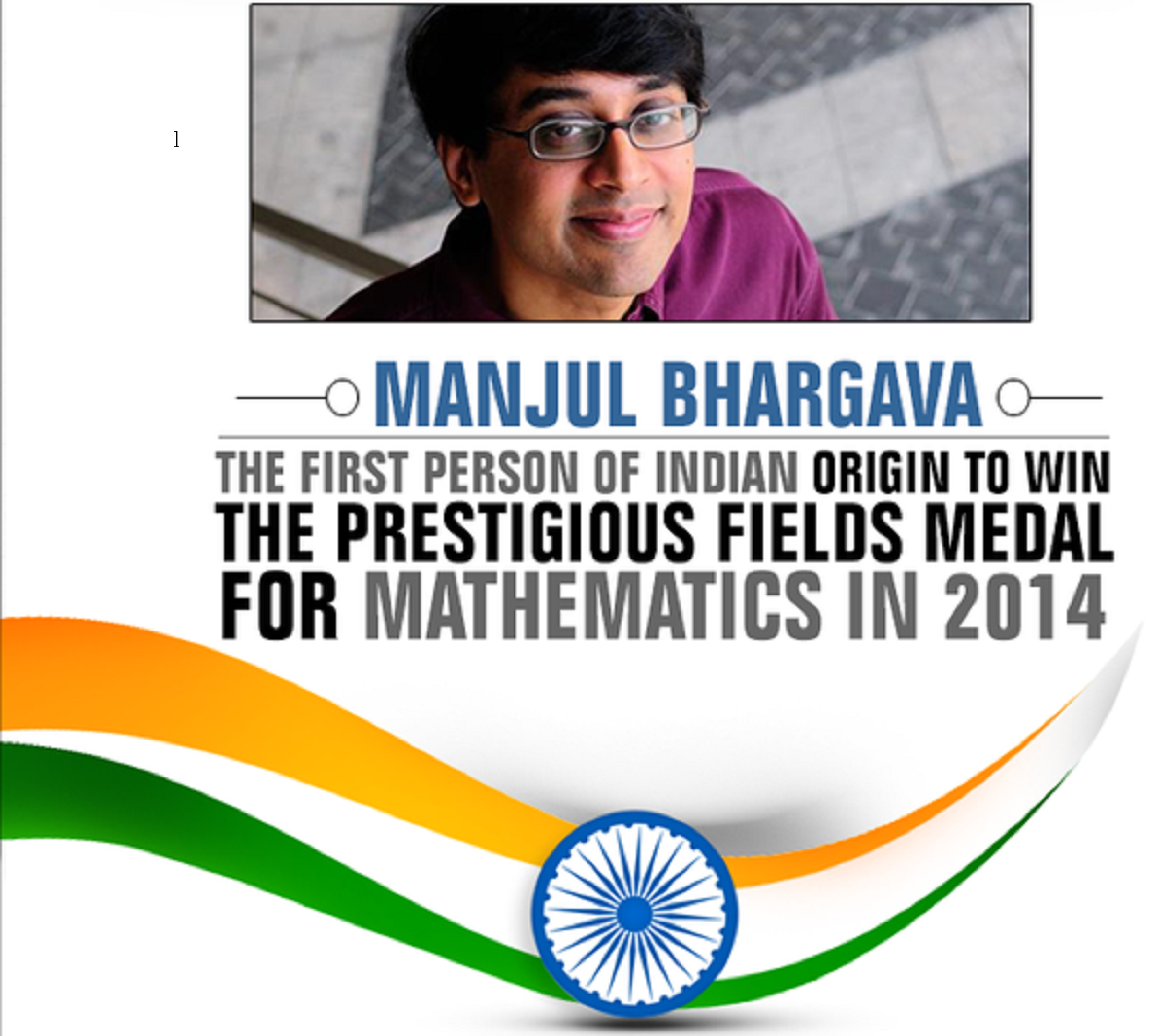
Lecture 1: **"Rajah Harish Chandra"** - Prof. Muruganandam V, NISER, Bhubaneshwar

Lecture 2: **"Contributions of Subbayya Sivasankaranarayana Pillai"** - Prof. Thangadurai R, HRI, Allahabad

Tour: All the participants were taken to Kumbakonam to visit **Ramanujan**'s house

1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